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Graph designs for the eight-edge five-vertex graphs

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ABSTRACT

The existence of graph designs for the two nonisomorphic graphs on five vertices and eight edges is determined in the case of index one, with three possible exceptions in total. It is established that for the unique graph with vertex sequence (3, 3, 3, 3, 4), a graph design of order n exists exactly when $n \equiv 0, 1 \pmod{16}$ and $n \neq 16$, with the possible exception of $n = 48$. For the unique graph with vertex sequence (2, 3, 3, 4, 4), a graph design of order n exists exactly when $n \equiv 0, 1 \pmod{16}$, with the possible exceptions of $n \in \{32, 48\}$.

1. Preliminaries

There are two nonisomorphic simple graphs on five vertices and eight edges, as shown in Fig. 1. The names G_{20} and G_{21} follow the numbering in [3].

Let $\mathcal{G}$ be a class of graphs. We consider decompositions both of complete graphs, and of complete graphs with specified 'missing' complete subgraphs, into copies of graphs in $\mathcal{G}$. To make this precise, for a given number m of vertices, when $m = \sum_{i=1}^{s} g_i u_i$, we write the partition of a set of size m into u_i classes of size g_i for $1 \leq i \leq s$ by the exponential notation $g_1^{u_1} \cdots g_s^{u_s}$. We call the partition sizes the *group type*. Now let $T = g_1^{u_1} \cdots g_s^{u_s}$ be a group type for order m. Then we denote by $G(T)$ the graph on m vertices obtained by first identifying a partition of type T of the vertices, calling the equivalence classes of the partition the *groups*. Then $G(T)$ contains precisely those edges whose endpoints are in different groups. Using graph theoretic nomenclature, $G(T)$ is a complete multipartite graph, with T representing the sizes of the classes in the partition.

A $\mathcal{G}$*-group divisible design* (*GDD* for short) of *type* T is a partition of all edges of $G(T)$ into graphs, so that each graph of the partition is isomorphic to a graph in the class $\mathcal{G}$. We permit group sizes to equal 0, and also the number u_i of groups of size g_i to be 0. We also permit that $g_i = g_j$ for $i \neq j$, so that 4^5 is the same as $4^4 4^1$, for example.

A $\mathcal{G}$-GDD of type 1^n is a $\mathcal{G}$*-design* or *graph design for* $\mathcal{G}$ *of order* n. See [4] for a summary of the main definitions and results. When $\mathcal{G} = \{K_k\}$, the notation of k-GDD is used; a k-GDD of type n^k is a *transversal design* $TD(k, n)$.

We give two examples. Table 1 gives a G_{20}-design of order 16 from [7]. Fig. 2 displays base graphs for a G_{20}-design of order 65, also from [7]. To obtain all 260 graphs, one applies addition modulo 65 to the vertex labels (see [6]).

Our interest is in G_{20}- and G_{21}-designs. These have arisen in a crucial manner in the problem of grooming traffic in optical networks [7], in addition to being among the smallest graphs for which existence of graph designs is not yet determined.

We assume familiarity with Wilson's fundamental construction (see [1,9] and references therein) in order to treat most of the exceptions in the existence spectrum. Indeed we settle the existence of G_{20}- and G_{21}-designs, leaving two possible exceptions (32 and 48) and one possible exception (48), respectively. These results are already included in the published reference [4] and the forthcoming survey [2], which base their reports on the results presented in this paper.

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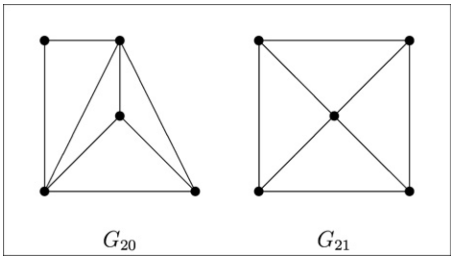

Fig. 1. The two graphs.

Table 1
Partition of K_{16} into G_{20}.

{0, 4}	{0, 5}	{0, 6}	{0, 7}	{4, 5}	{4, 6}	{4, 7}	{5, 6}
{0, 8}	{0, 9}	{0, 10}	{0, 11}	{8, 9}	{8, 10}	{8, 11}	{9, 10}
{0,12}	{0, 13}	{0, 14}	{0, 15}	{12, 13}	{12, 14}	{12, 15}	{13, 14}
{1, 2}	{1, 0}	{1, 3}	{1, 4}	{2, 0}	{2, 3}	{2, 4}	{0, 3}
{1, 5}	{1, 7}	{1, 8}	{1, 9}	{5, 7}	{5, 8}	{5, 9}	{7, 8}
{1, 6}	{1, 10}	{1, 12}	{1, 13}	{6, 10}	{6, 12}	{6, 13}	{10, 12}
{14, 15}	{14, 1}	{14, 11}	{14, 4}	{15, 1}	{15, 11}	{15, 4}	{1, 11}
{2, 5}	{2, 11}	{2, 12}	{2, 13}	{5, 11}	{5, 12}	{5, 13}	{11, 12}
{2, 14}	{2, 7}	{2, 10}	{2, 9}	{14, 7}	{14, 10}	{14, 9}	{7, 10}
{6, 8}	{6, 2}	{6, 15}	{6, 14}	{8, 2}	{8, 15}	{8, 14}	{2, 15}
{3, 5}	{3, 10}	{3, 15}	{3, 14}	{5, 10}	{5, 15}	{5, 14}	{10, 15}
{3, 6}	{3, 7}	{3, 11}	{3, 9}	{6, 7}	{6, 11}	{6, 9}	{7, 11}
{3, 8}	{3, 4}	{3, 12}	{3, 13}	{8, 4}	{8, 12}	{8, 13}	{4, 12}
{4, 11}	{4, 10}	{4, 13}	{4, 9}	{11, 10}	{11, 13}	{11, 9}	{10, 13}
{7, 9}	{7, 13}	{7, 15}	{7, 12}	{9, 13}	{9, 15}	{9, 12}	{13, 15}

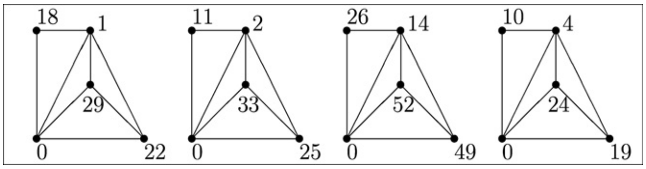

Fig. 2. Partition of K_{65} into G_{20}.

2. The existence spectrum for G_{20}

Since G_{20} has eight edges, simple counting establishes that for a G_{20}-design of order v to exist, it must be the case that $\binom{v}{2} \equiv 0 \pmod 8$, which in turn requires that $v \equiv 0, 1 \pmod{16}$. In 1980 Bermond et al. [3] produced G_{20}-designs for orders 17, 33, 49, 97, 113, and 177. Using these results, Rodger [10] established existence when $v \equiv 1 \pmod{16}$ except possibly when $v = 65$. Colbourn and Wan [7] then settled existence when $v = 65$ (see Fig. 2), and Chang [5] independently settled the same case somewhat later. Thus, existence when $v \equiv 1 \pmod{16}$ is completely settled. The case when $v \equiv 0 \pmod{16}$ has a more chequered history. An incomplete statement in [10] is used in [5] to assert that existence is settled in this case as well, but in fact prior to [7] in 2001, not a single G_{20}-design of even order was published. To date, the only published example is the G_{20}-design of order 16 in Table 1, taken from [7].

Nevertheless, substantial partial results are known:

Lemma 2.1 ([7]). *There exist G_{20}-GDDs of types 4^5, 4^7, 4^9, and 4^{11}.*

Lemma 2.2. *There exists a G_{20}-GDD of type 4^{13}.*

Proof. Using the three base graphs in Fig. 3, apply addition in $\mathbb{Z}_{52}$ to form 156 graphs. These partition all have edges except those whose endpoints differ by 13 or 26 modulo 52. □

In the same manner, Figs. 4–6 establish

Lemma 2.3. *There exist G_{20}-GDDs of types 16^4, 16^8, and 16^{12}.*

Lemma 2.4. *For $n \geq 4$, there exists a G_{20}-GDD of type 16^n.*

Proof. When a G_{20}-GDD of type 4^n exists, using Wilson's fundamental construction with weight four, produces a G_{20}-GDD of type 16^n; this handles $n \in \{5, 7, 9, 11, 13\}$ by Lemmas 2.1 and 2.2. Lemma 2.3 handles $n \in \{4, 8, 12\}$. Next we employ

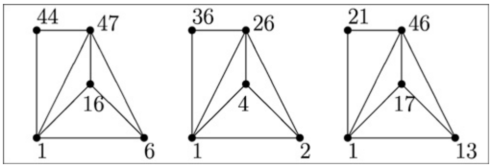

Fig. 3. Type 4^{13} on $\mathbb{Z}_{52}$.

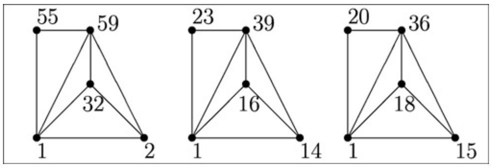

Fig. 4. Type 16^4 on $\mathbb{Z}_{64}$.

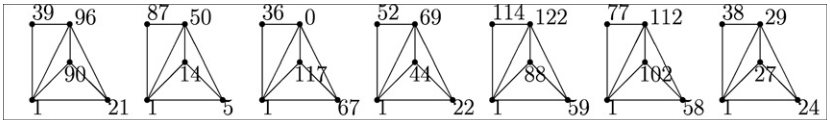

Fig. 5. Type 16^8 on $\mathbb{Z}_{128}$.

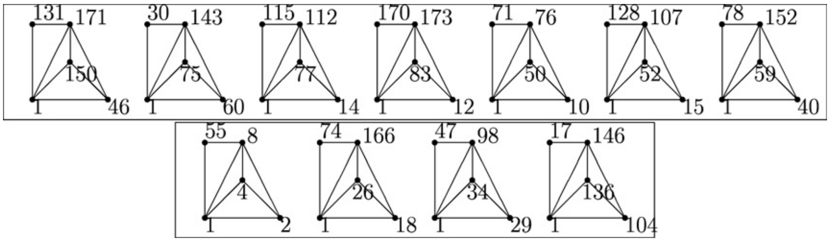

Fig. 6. Type 16^{12} on $\mathbb{Z}_{192}$.

existence results on $\{5, 9, 13\}$-GDDs of type 4^n from [1]. Whenever a 5-GDD of type 4^n exists, using the G_{20}-GDD of type 4^5 we obtain a G_{20}-GDD of type 16^n. Thus all cases when $n \equiv 0, 1 \pmod 5$ and $n \geq 5$ are handled. In the same manner when $n \notin \{2, 4, 7, 9, 12\}$ and $n \equiv 2, 4 \pmod 5$, there is a $\{5, 9\}$-GDD of type 4^n and hence a G_{20}-GDD of type 16^n. Finally for $n \equiv 3 \pmod 5$ and $n \geq 18$, there is a $\{5, 13\}$-GDD of type 4^n and hence a G_{20}-GDD of type 16^n. □

Theorem 2.5. *There exists a G_{20}-design of order $16n$ for every positive integer n except possibly when $n \in \{2, 3\}$.*

Proof. When $n = 1$, use Table 1. When $n \geq 4$, apply Lemma 2.4 and fill each group with a G_{20}-design of order 16. The designs on 32 and 48 points are possible exceptions. □

3. The existence spectrum for G_{21}

Once again, the necessary condition is that $v \equiv 0, 1 \pmod{16}$. Bermond et al. [3] settle all cases affirmatively when $v \equiv 1 \pmod{16}$, and give a G_{21}-design of order 64. They establish that there is no G_{21}-design of order 16. There has been no progress since. The lack of a solution on 16 points prevents us from effective application of the type of argument used for G_{20}. Our general strategy is to embed large subdesigns from the $1 \pmod{16}$ class into designs in the $0 \pmod{16}$ class. First we make a useful observation.

Lemma 3.1. *There exist G_{21}-GDDs of types 4^4 and $4^3 12^1$.*

Proof. The G_{21}-GDD of type 4^4 is from [7]. There is a resolvable P_3-GDD of type 4^3 having six parallel classes. Extend each parallel class using two infinite points to obtain the G_{21}-GDD of type $4^3 12^1$. □

Lemma 3.2. *There exists a G_{21}-design of order 32.*

Proof. We produce a solution on $(\mathbb{Z}_5 \times \{0, 1, 2, \alpha, \beta\}) \cup \{\infty_i : 0 \leq i < 7\}$. First place a G_{21}-design of order 17 on $(\mathbb{Z}_5 \times \{\alpha, \beta\}) \cup \{\infty_i : 0 \leq i < 7\}$. Now develop the base blocks shown in Fig. 7 under the action of $\mathbb{Z}_5$, fixing ∞ and fixing subscripts. □

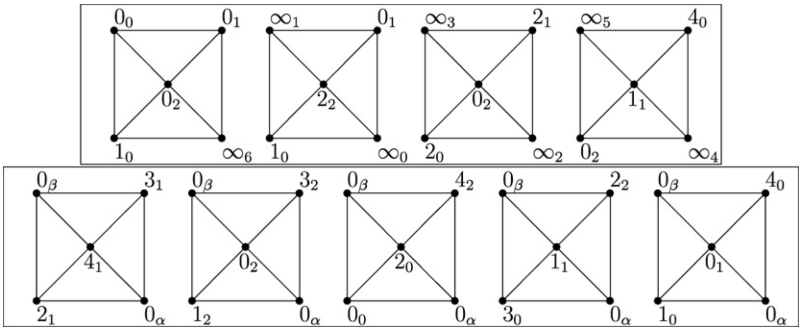

Fig. 7. The G_{21}-design of order 32.

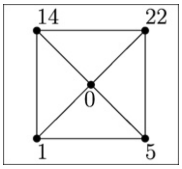

Fig. 8. Type 8^3 on $\mathbb{Z}_{24}$.

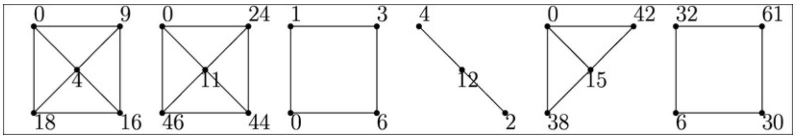

Fig. 9. Base blocks for order 80 over $\mathbb{Z}_{63}$.

Lemma 3.3. *There exists a G_{21}-GDD of type 8^3.*

Proof. Use the base block in Fig. 8. $\square$

Lemma 3.4. *When $v \equiv 0 \pmod{32}$, there exists a G_{21}-design of order v.*

Proof. Employ a 3-GDD of type $4^{v/32}$ when $v \equiv 0, 32 \pmod{96}$, or $4^{(v-64)/32}8^1$ when $v \equiv 64 \pmod{96}$; inflate using the G_{21}-GDD of type 8^3 and fill in holes using the G_{21}-designs of orders 32 and 64. $\square$

In order to treat the class when $v \equiv 16 \pmod{32}$, we require some direct constructions. As in the case of $v = 32$, we assume the presence of a subdesign whose order is $w \equiv 1 \pmod{16}$ for which $x = v - w \equiv 3 \pmod 6$; but in these cases we always employ a *cyclic* automorphism on the x points. To shorten the proofs, we establish a useful technical lemma.

Lemma 3.5. *Let $V = \mathbb{Z}_{6w+3}$ and let D be a subset of even cardinality of $\{d : 1 \le d \le 3w+1 : d \not\equiv 0 \pmod 3\}$. Let G be the circulant graph on V in which the edges are $\{\{i, i+d\} : i \in \mathbb{Z}_{6w+3}, d \in D\}$. Then G has a resolvable decomposition into paths on two edges; the number of parallel classes is $\frac{3}{2}|D|$.*

Proof. Form D' from D by replacing every $d \in D$ for which $d \equiv 2 \pmod 3$ by $6w+3-d$. Now write $D' = \{d_1, \dots, d_{2s}\}$. For $1 \le i \le s$, form the path with edges $\{0, d_{2i}\}$ and $\{d_{2i}, d_{2i} + d_{2i-1} \bmod 6w+3\}$. Repeatedly adding 3 (modulo $6w+3$) produces one parallel class of paths. Adding 1 to each vertex produces a second, and adding 2 produces a third. $\square$

Lemma 3.5 is used in the constructions which follow, to treat $2s$ differences on the x points while exhausting all edges involving $3s$ of the w points, and so we need not explicitly comment on the G_{21}s that arise in this manner. Now we produce the solution for $v = 80$.

Lemma 3.6. *There is a G_{21}-design of order 80.*

Proof. We employ the six base graphs in Fig. 9. Develop each modulo 63. The first two each produce copies of G_{21}. Now treat the third and fourth together. Considering the neighbors of the (missing) central vertex that was deleted to form the cycle of length four, along with the three elements on the diagonal path in the fourth graph, we find that the seven labels (0, 1, 2, 3, 4, 12, 6) are distinct modulo 7. Add an infinite point α_0 as the central vertex for the cycle and the lower left corner for the path and, as these two graphs are developed by addition of i modulo 63, use the vertex $\alpha_{i \bmod 7}$ in place of α_0. In this way, seven infinite points are added; the third graph now generates G_{21}s, while the fourth still only generates $K_4 - e$s. To

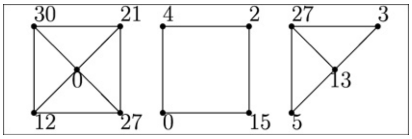

Fig. 10. Base blocks for order 112 over $\mathbb{Z}_{63}$.

complete them, observe that $(12, 4, 2) \equiv (0, 1, 2) \pmod 3$. Hence, the copies of $K_4 - e$ modulo 63 can be partitioned into three classes so that within each class the 21 diagonal paths are disjoint. Adjoin three new infinite points to complete these to copies of G_{21}. For the fifth and sixth base graph, we proceed similarly by noting that (42, 15, 30, 38, 32, 61, 6) are distinct modulo 7, and hence can be completed to G_{21}s using seven further infinite points. In this way, seventeen infinite points have been added in total, and we place a G_{21}-design of order 17 on them. □

Lemma 3.7. *There exist G_{21}-designs of orders* 176, 208*, and* 240*.*

Proof. Start with a TD(4, 8). Give weight 4 in three groups and weights 4 and 12 in the fourth (using G_{21}-GDDs of types 4^4 and $4^3 12^1$) to form a G_{21}-GDD of type $32^3 80^1$; fill groups to treat order 176. For order 208, first use TD(3, 16) and inflate giving weight 2 in two groups and 1 in the third; use a single G_{21}, which is itself a G_{21}-GDD of type $2^2 1^1$ in the inflation. The result is a G_{21}-GDD of type $32^2 16^1$. Now use a TD(3, 8) and give weight 8 using the G_{21}-GDD of type 8^3 to form a G_{21}-GDD of type 64^3. Adjoin sixteen new points. Then for two groups of the G_{21}-GDD of type 64^3, place on the group and the sixteen new points, a copy of the G_{21}-GDD of type $32^2 16^1$, aligning the group of size 16 with the sixteen new points. The result is a G_{21}-GDD of type $32^4 80^1$. Fill groups to get the G_{21}-design of order 208. For order 240, use a TD(3, 10) and give weight 8 using the G_{21}-GDD of type 8^3 to form a G_{21}-GDD of type 80^3; fill its groups. □

Lemma 3.8. *There is a G_{21}-design of order* 112*.*

Proof. We employ the three base graphs in Fig. 10. Develop each modulo 63. The first produces copies of G_{21}. For the second and third base graphs, we proceed by noting that (0, 15, 2, 3, 4, 5, 13) are distinct modulo 7, and hence can be completed to G_{21}s using seven infinite points. In total 49 infinite points are needed, and these are provided by 21 further parallel classes of paths whose existence is ensured by Lemma 3.5 (all differences that are multiples of three are consumed in Fig. 10). Finally place a G_{21}-design of order 49 on the infinite points. □

Lemma 3.9. *There is a G_{21}-design of order* 144*.*

Proof. Again we work modulo 63 and adjoin 81 infinite points using a G_{21}-design of order 81. First form the base graph on {0, 5, 7, 21} missing (only) the edge {5, 7}; because (0, 7, 5) are distinct modulo 3, we can employ three infinite points to extend these to copies of G_{21}. Next we form seven paths as follows: (0, 30, 27), (1, 25, 19), (2, 14, 5), (3, 18, 36), (7, 34, 53), (8, 10, 20), (12, 16, 17). The 21 vertices named are distinct modulo 21, and hence these form a set of 21 parallel classes of paths. Use 42 infinite points to complete these to copies of G_{21}. There remain 12 differences, none a multiple of 3. Apply Lemma 3.5 to form 18 parallel classes of paths; then add 36 more infinite points, for a total of 81. □

Having presented solutions whenever $64 \le v < 256$, we are in a position to complete the proof for larger orders.

Theorem 3.10. *Let $v \equiv 0 \pmod{16}$ and $v \ge 256$. Then there exists a G_{21}-design of order v.*

Proof. Write $m = \lfloor v/64 \rfloor$ and $x = (v - 64m)/16$, so that $v = 64m + 16x$, $0 \le x \le 2m$, and $m \ge 4$. (In fact we always have $0 \le x \le 3$ and hence the inequality that $0 \le x \le 2m$ is immediate.) Then there exists a TD(4, 4m). Give weight 4 in three of the groups, and weight 12 to $2x$ points and weight 4 to $4m - 2x$ points in the final group, using G_{21}-GDDs of types 4^4 and $4^3 12^1$ from Lemma 3.1. The result is a G_{21}-GDD of type $(16m)^3(16m + 16x)^1$. A simple induction using the designs of size less than v (and at least 64) provides the designs needed to fill the holes to produce the G_{21}-design of order v. □

This leads to the current existence theorem:

Theorem 3.11. *A G_{21}-design of order v exists if, and only if, $v \equiv 0, 1 \pmod{16}$, except when $v = 16$ and possibly when $v = 48$.*

4. Conclusions

The existence of graph designs for small graphs is a difficult problem, primarily because the solutions for "small" cases are often difficult to produce by computational exhaustive techniques, yet finding appropriate structure to make the search feasible is not well understood. In this paper, most of the progress results from two ideas. The first is the use of $\mathcal{G}$-GDDs, and is by now standard. The second is the use of subdesigns of order 1 modulo 16 to produce designs in the 0 mod 16 class, and in particular the use of relatively large subdesigns. Neither technique seems appropriate to settle the remaining cases, and hence they remain as possible exceptions.

Next, we remark that these results have application to the grooming problem for optical networks, and refer the reader to [7] for more details.

Finally, we conclude that the elementary necessary conditions for the existence of five-vertex graphs are also sufficient except as follows (see [2,3,5,8]), where the graph numbers are those from [4]:

Exceptions:
$(n, G) \in \{(5, G_7), (5, G_8), (5, G_9), (6, G_9), (9, G_{14}), (12, G_{14}), (7, G_{16}), (8, G_{16}), (8, G_{18}), (14, G_{18}), (8, G_{19}), (16, G_{21}), (9, G_{22}), (10, G_{22}), (18, G_{22})\}$.

Unresolved Cases:
$(n, G) \in \{(32, G_{20}), (48, G_{20}), (48, G_{21}), (27, G_{22}), (36, G_{22}), (54, G_{22}), (64, G_{22}), (72, G_{22}), (81, G_{22}), (90, G_{22}), (135, G_{22}), (144, G_{22}), (162, G_{22}), (216, G_{22}), (234, G_{22})\}$.

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